

p -adic unit roots of L -functions over finite fields

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Abstract

In this brief note, we consider p -adic unit roots or poles of L -functions of exponential sums defined over finite fields. In particular, we look at the number of unit roots or poles, and a congruence relation on the units. This raises a question in arithmetic mirror symmetry.

1 Introduction

L -functions of exponential sums over finite fields are rational functions. The following note considers two questions concerning the p -adic unit roots and poles of these rational functions. First, how many p -adic unit roots and poles are there? Second, what type of geometric relation do the unit roots satisfy; more specifically, is there a congruence relation the unit roots satisfy?

To motivate, consider the affine Legendre family of elliptic curves $E_{\bar{\lambda}}$ defined by $y^2 = x(x-1)(x-\bar{\lambda})$, where $\bar{\lambda} \in \mathbb{F}_p \setminus \{0, 1\}$. Setting

$$\#E_{\bar{\lambda}}(\mathbb{F}_{p^m}) := \#\{(\bar{x}, \bar{y}) \in \mathbb{F}_{p^m}^2 \mid \bar{y}^2 = \bar{x}(\bar{x}-1)(\bar{x}-\bar{\lambda})\},$$

then it is well-known that the associated zeta function takes the form

$$Z(E_{\bar{\lambda}}/\mathbb{F}_p, T) := \exp\left(\sum_{m=1}^{\infty} \#E_{\bar{\lambda}}(\mathbb{F}_{p^m}) \frac{T^m}{m}\right) = \frac{(1 - \pi_0(\bar{\lambda})T)(1 - \pi_1(\bar{\lambda})T)}{(1 - pT)}.$$

Denote by $H(\lambda)$ the series obtained from the hypergeometric series ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \lambda)$ by taking the terms of λ up to degree $(p-1)/2$. If $H(\bar{\lambda}) \neq 0$, then there is a unique p -adic unit root $\pi_0(\bar{\lambda})$ of the zeta function, and

$$\pi_0(\bar{\lambda}) \equiv (-1)^{(p-1)/2} H(\bar{\lambda}) \pmod{p}. \quad (1)$$

In this example, the number of unit roots is bounded by the genus 1. For hyperelliptic curves of genus g , the number of p -adic unit roots is bounded by the genus g . The geometric interpretation of (1) comes from noting that the hypergeometric series ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \lambda)$ is the unique holomorphic solution of the associated Picard-Fuchs equation of the Legendre family. Formulas for other families have also been recently studied; see for instance [3] and [36].

In the following, we focus on L -functions of exponential sums over a finite field, defined over hypersurfaces in the algebraic torus \mathbb{G}_m^n or affine space. The main results are Theorems 2.1, 3.1, and 4.1. Section 2 considers rational functions in one-variable, and it is shown that in many cases which arise in practice, the unit roots are in fact 1-units, meaning

they satisfy $|x - 1|_p < 1$. Since this is not always the case, the result is then generalized to all rational functions. Sections 3 and 4 consider L -functions in many variables defined over affine hypersurfaces in \mathbb{G}_m^n and \mathbb{A}^n , respectively. Lastly, motivated from the result in Section 3, Section 5 looks at the question of comparing the L -functions of exponential sums of regular functions defined on the complement of two mirror varieties.

We note that similar questions may be asked for other types of zeta functions, such as Dwork's unit root L -function [33] [28] [27] [17] or L -functions over rings, such as [6]. This has already been answered in the case of the zeta function of divisors [16] [18] [30], and so it would be interesting to study similar questions for the function field height zeta function [15] [29].

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2 Rational functions in one variable

Let \mathbb{F}_q be the finite field with q elements and characteristic p , and ψ a non-trivial additive character of \mathbb{F}_q . Set $\psi_m := \psi \circ \text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$. Let $f/g \in \mathbb{F}_q(x)$ be a rational function. Throughout this section, we will assume f/g has a pole at zero and infinity (meaning, $g(0) = 0$ and $\deg(f) > \deg(g)$), and the order of every pole is not divisible by p . For each $m \geq 1$, define the exponential sum

$$S_m(f/g) := \sum_{\bar{x} \in \mathbb{F}_{q^m}, g(\bar{x}) \neq 0} \psi_m(f(\bar{x})/g(\bar{x})).$$

It follows from [26] (or [22]) that the associated L -function

$$L(f/g, T) := \exp \left(\sum_{m=1}^{\infty} S_m(f/g) \frac{T^m}{m} \right)$$

is a polynomial over $\mathbb{Z}[\zeta_p]$ of degree $\deg(f) + n - 1$ with precisely n p -adic unit roots, where ζ_p is a primitive p -th root of unity. When the roots of g lie in the base field \mathbb{F}_q , then each of the units of the L -function must be a 1 -unit, or *principal unit*, as we will now show; this means they satisfy $|x - 1|_p < 1$.

Theorem 2.1. *Assuming $p > n$ and that every root of g lies in \mathbb{F}_q , then each p -adic unit root of $L(f/g, T)$ is a 1 -unit.*

Proof. Let η_1, \dots, η_n denote the unit roots of $L(f/g, T)$. Let $\tilde{\pi}$ be the uniformizer of the ring of integers of the field obtained by adjoining the zeros of $L(f/g, T)$ to $\mathbb{Q}_p(\zeta_p)$. Since $L(f/g, T) = \prod (1 - \eta_j T) \pmod{\tilde{\pi}}$, we see that $S_m(f/g) \equiv -\sum \eta_j^m \pmod{\tilde{\pi}}$. On the other hand, since $\zeta_p \equiv 1 \pmod{\tilde{\pi}}$, we have $S_m(f/g) \equiv -n \pmod{\pi}$. Consequently, the power sums $p_m := \sum \eta_j^m \equiv n \pmod{\tilde{\pi}}$ for every m .

Consider now the product $\prod_{j=1}^n (1 - \eta_j) = \sum_{k=0}^n (-1)^k e_k$, where e_k is the k -th elementary symmetric polynomial in η_1, \dots, η_n . Newton's identity relates e_k and the power sums p_k :

$$e_1 = p_1 \quad \text{and} \quad k e_k = \sum_{i=1}^k (-1)^{i+1} e_{k-i} p_i.$$

In our case, since $p > n$ and $p_m \equiv n \pmod{\tilde{\pi}}$, we see that $e_k \equiv \binom{n}{k}$ for each $k = 0, 1, \dots, n$. Hence

$$\prod_{j=1}^n (1 - \eta_j) \equiv \sum_{k=0}^n (-1)^k \binom{n}{k} = (1 - 1)^n = 0 \pmod{\tilde{\pi}},$$

implying at least one of the unit roots must be a 1-unit. If we suppose $\eta_1 \equiv 1 \pmod{\tilde{\pi}}$, then we get the new power sum identity $\sum_{j=2}^n \eta_j^m \equiv n-1 \pmod{\tilde{\pi}}$. Repeating the above argument obtains the result. \square

We note that, when the roots of g lie in the base field \mathbb{F}_q , then Zhu [37] has shown that the other roots have slopes lying on or above the Hodge polygon.

While unit roots of L -functions need not always be 1-units, a similar statement to Theorem 2.1 for general one-variable rational functions does hold. To state this, define $Z(g, T) := \exp\left(\sum N_m \frac{T^m}{m}\right)$, the zeta function associated with the point count

$$N_m := \#\{\bar{x} \in \mathbb{F}_{q^m} \mid g(\bar{x}) = 0\}.$$

Theorem 2.2. *Let d_i be the degrees of the irreducible polynomials factoring g over \mathbb{F}_q . If $p > \deg(f) + n - 1$, then*

$$L(f/g, T) \equiv \prod (1 - T^{d_i}) \pmod{\pi},$$

where $\pi := 1 - \zeta_p$.

Proof. Factoring g into irreducibles over \mathbb{F}_q of degrees d_i , note that $Z(g, T)^{-1} = \prod (1 - T^{d_i})$. Next, observe that $S_m(f/g) \equiv -N_m \pmod{\pi}$ for every m . Since p is greater than the degrees of $L(f/g, T)$ and $Z(g, T)^{-1}$, the result follows. \square

3 Complements of hypersurfaces in \mathbb{G}_m

We now generalize Theorem 2.2 to many variables. Let $f, g \in \mathbb{F}_q[x_1, \dots, x_n]$ be polynomials of degree D and d , respectively. Let H_g^* denote the hypersurface in \mathbb{G}_m^n defined by $g = 0$ over \mathbb{F}_q , and let V_g^* denote the complement of H_g^* in \mathbb{G}_m^n . For each positive integer m define the exponential sum

$$S_m(V_g^*, f) = \sum \psi_m(f(x)),$$

where the sum runs over all $x \in V_g^*(\mathbb{F}_{q^m})$. The associated L -function

$$L(V_g^*, f; T) := \exp\left(\sum_{m=1}^{\infty} S_m(V_g^*, f) \frac{T^m}{m}\right)$$

is well-known to be a rational function defined over the cyclotomic field $\mathbb{Q}(\zeta_p)$, where ζ_p is a primitive p -th root of unity. Denote by \hat{g} the polynomial of obtained from g by replacing the coefficients with their Teichmüller representative. Decompose $\hat{g} = \hat{g}_d + \hat{g}_{d-1} + \dots + \hat{g}_0$ into homogeneous forms \hat{g}_i of degree i . Throughout this section, we will assume that

$$\hat{g}_d \text{ is a product of distinct irreducible factors.} \tag{2}$$

This condition is not required if $n = 1$.

Let $Z(H_g^*, T) := \exp\left(\sum N_m \frac{T^m}{m}\right)$ be the zeta function associated with the point count

$$N_m := \#H_g^*(\mathbb{F}_{q^m}) := \#\{x \in (\mathbb{F}_{q^m})^n \mid g(x) = 0\}.$$

As a consequence of the Dwork trace formula, the zeta function has the following simple description mod p (see [32]).

Define the \mathbb{F}_q -vector space

$$R_d := \mathbb{F}_q[x_1, \dots, x_n]_{\leq d}$$

consisting of polynomials over \mathbb{F}_q of degree at most d . Define the Cartier operator ψ_q as follows. Writing $x^u := x_1^{u_1} \cdots x_n^{u_n}$, then

$$\psi_q(x^u) := \begin{cases} x^{u/q} & \text{if } q \mid u_i \text{ for every } i \\ 0 & \text{otherwise.} \end{cases}$$

Observe that multiplication by g^{q-1} is a map from R_d into R_{dq} , and thus $\psi_q \circ g^{q-1}$ is an operator on R_d . Wan [32] observes that

$$Z(H_g^*, T)^{(-1)^n} \equiv (1 - T) \det(1 - \psi_q \circ g^{q-1} T \mid R_d) \pmod{p}, \quad (3)$$

a polynomial of degree at most $1 + \dim_{\mathbb{F}_q} R_d = 1 + \binom{n+d}{n}$.

Theorem 3.1. *Suppose (2) and $p > \max\{\binom{n+d}{n} + 1, D\}$. Then*

$$L(V_g^*, f; T) \equiv (1 - T)^{(-1)^n} \cdot Z(H_g^*, T)^{-1} \pmod{\pi}$$

where $\pi := 1 - \zeta_p$. Moreover,

$$L(V_g^*, f; T)^{(-1)^{n+1}} \equiv \det(1 - \psi_q \circ g^{q-1} T \mid R_d) \pmod{\pi},$$

a polynomial of degree at most $\binom{d+n}{n}$. In particular, the number of unit roots of $L(V_g^*, f; T)$ is bounded by $\binom{d+n}{n}$.

Proof. From the Reich trace formula [25],

$$L(V_g^*, f; T)^{(-1)^{n+1}} = \det(1 - \alpha T)^{\delta^n} = \frac{\prod (1 - \omega_i T)}{\prod (1 - \eta_j T)},$$

where α is a p -adic operator acting on a space of functions, and δ is the map defined by $h(T)^\delta := h(T)/h(qT)$. Hence, $L(V_g^*, f; T)^{(-1)^{n+1}} \equiv \det(1 - \alpha T) \pmod{q}$. We refine this further using the work of Adolphson-Sperber [1], which says that the q -adic Newton polygon of $\det(1 - \alpha T)$ lies on or above the lower convex hull of the points $(0, 0)$ and $(\sum_{k=0}^m W(k), D^{-1} \sum_{k=0}^m kW(k))$ for $m = 0, 1, 2, \dots$. We refer the reader to [1] for the definition of $W(k)$ as it is complicated; however, we do note some properties. First, $W(k) \geq \binom{n+k-1}{k}$ for $k \geq 0$, and so $W(k) > 0$ for $k \geq 0$. Second $W(0) = \binom{n+d}{n}$.

By hypothesis $p > D$, and so $\text{ord}_\pi(\cdot) \geq (p-1)\text{ord}_q(\cdot) \geq D\text{ord}_q(\cdot)$. Consequently, $\det(1 - \alpha T) \pmod{\pi}$ is a polynomial of degree at most $W(0)$. Next, since $\zeta_p \equiv 1 \pmod{\pi}$ and $\#V_g^*(\mathbb{F}_{q^m}) \equiv (-1)^n - \#H_g^*(\mathbb{F}_{q^m}) \pmod{q}$, we have

$$(-1)^{n+1} S_m(V_g^*, f) \equiv (-1)^{n+1} \#V_g^*(\mathbb{F}_{q^m}) \equiv -1 + (-1)^n \#H_g^*(\mathbb{F}_{q^m}) \pmod{\pi}. \quad (4)$$

By (3), $Z(H_g^*, T)^{(-1)^n} \pmod{p}$ is a polynomial of degree at most $\binom{d+n}{n}$. Since p is strictly larger than the maximum degrees $W(0)$ and $\binom{d+n}{n}$, equation (4) implies

$$L(V_g^*, f; T)^{(-1)^{n+1}} \equiv (1 - T)^{-1} \cdot Z(H_g^*, T)^{(-1)^n} \equiv \det(1 - \psi \circ g^{q-1} T \mid R_d) \pmod{\pi}.$$

□

Remarks.

1. Toric exponential sums are those defined over the algebraic torus \mathbb{G}_m^n . In this case, $g(x) = x_1 \cdots x_n$, and the above shows $L(\mathbb{G}_m^n, f; T)^{(-1)^{n+1}}$ has a unique p -adic unit root. Adolphson and Sperber [3] have demonstrated that this unit root has a formula similar to (1) using \mathcal{A} -hypergeometric functions.
2. It is quite likely that improvements may be made weakening the condition $p > \max\{\binom{n+d}{d} + 1, D\}$. More interesting would be to obtain formulas for higher congruences.
3. Through personal communication, Alan Adolphson suggests that $\binom{n+d}{d}$ is generically the precise number of unit roots of $L(V_g^*, f; T)^{(-1)^{n+1}}$. This would likely take the form of an Hasse polynomial (see, for example, [7]).

4 Complements of hypersurfaces in affine space

Let $f, g \in \mathbb{F}_q[x_1, \dots, x_n]$ be polynomials of degree D and d , respectively. Let H_g be the variety in \mathbb{A}^n defined by $g = 0$ over \mathbb{F}_q . Denote by V_g the complement of H_g in \mathbb{A}^n . For each positive integer m define the exponential sum

$$S_m(V_g, f) = \sum \psi_m(f(x)),$$

where the sum runs over all $x \in V_g(\mathbb{F}_{q^m})$. The associated L -function

$$L(V_g, f; T) := \exp \left(\sum_{m=1}^{\infty} S_m(V_g, f) \frac{T^m}{m} \right)$$

is a rational function defined over the cyclotomic field $\mathbb{Q}(\zeta_p)$, where ζ_p is a primitive p -th root of unity.

As in the previous section, let \hat{g} denote the Teichmüller lift of g . Set $S := \{1, 2, \dots, n\}$. For a subset $J \subset S$, denote by \hat{g}_J the polynomial obtained from \hat{g} by setting $x_i = 0$ for every $i \in J$. Throughout this section, we will assume that

for every $J \subset S$, the highest degree homogenous form of \hat{g}_J is a product of distinct irreducible factors. (5)

Let $Z(H_g, T) := \exp \left(\sum N_m \frac{T^m}{m} \right)$ be the zeta function associated with the point count

$$N_m := \#H_g(\mathbb{F}_{q^m}) := \#\{x \in (\mathbb{F}_{q^m})^n \mid g(x) = 0\}.$$

As a consequence of the Dwork trace formula, the zeta function has the following simple description mod p (see [32]).

Define the \mathbb{F}_q -vector space

$$W_d := (x_1 \cdots x_n \mathbb{F}_q[x_1, \dots, x_n])_{\leq d}$$

consisting of polynomials over \mathbb{F}_q of degree at most d which are divisible by the monomial $x_1 \cdots x_n$. Observe that multiplication by g^{q-1} is a map from W_d into W_d , and thus $\psi_q \circ g^{q-1}$ is an operator on W_d . Wan [32] shows that

$$Z(H_g, T)^{(-1)^n} \equiv \det(1 - \psi_q \circ g^{q-1} T \mid W_d) \pmod{p}, \quad (6)$$

a polynomial of degree at most $\dim_{\mathbb{F}_q} W_d = \binom{d}{n}$. Observe that the Chevalley-Warning result $\#V_g(\mathbb{F}_q) \equiv 0 \pmod{p}$ follows if $n > d$ since $W_d = 0$ in this case.

Theorem 4.1. *Suppose (5) and $p > \max\{\binom{n+d}{d} + 1, D\}$. Then*

$$L(V_g, f; T) \equiv Z(H_g, T)^{-1} \pmod{\pi}$$

where $\pi := 1 - \zeta_p$. Moreover,

$$L(V_g, f; T)^{(-1)^{n+1}} \equiv \det(1 - \psi_q \circ g^{q-1} T \mid W_d) \pmod{\pi},$$

a polynomial of degree at most $\binom{d}{n}$. In particular, the number of unit roots of $L(V_g, f; T)$ is bounded by $\binom{d}{n}$.

Proof. The toric decomposition of affine space implies

$$L(V_g, f; T) = \prod_{J \subset S} L(V_{g_J}^*, f_J; T).$$

Thus, by Theorem 3.1,

$$L(V_g, f; T) \equiv \prod_{J \subset S} Z(H_{g_J}^*, T)^{-1} \equiv Z(H_g, T)^{-1} \pmod{\pi}$$

The rest of the result follows from (6). □

An immediate corollary of the above is the following Chevalley-Warning type result:

Corollary 4.2. *Under the same conditions as Theorem 4.1, suppose further that $n > d$. Then $\text{ord}_q S_1(V_g, f) > 0$.*

Theorem 4.1 and Corollary 4.2 raise the following:

Questions and Remarks.

1. Observe that for a parametrized family of hypersurfaces g_λ and regular functions f_λ on V_{g_λ} , dependent on an algebraic parameter λ , it is expected that the unit roots of $L(V_{g_\lambda}, f_\lambda; T)$ modulo an appropriate uniformizer will only depend on the family g_λ . For example, setting $g_\lambda = y^2 - x(x-1)(x-\lambda)$, the Legendre family of (affine) elliptic curves mentioned in the introduction, then by (4.1) the $L(V_{g_\lambda}, f_\lambda; T)^{-1}$ will have a unique unit root $\pi_0(\lambda)$ satisfying (1) when $H(\lambda) \neq 0$.
2. When does $L(V_g, f; T)$, or equivalently, $Z(V_g, T)$, only have unit roots which are 1-units? This will be the case when $\det(1 - \psi_q \circ g^{q-1} T)$ equals $(1 - T)^e$ for some $e \leq \binom{d}{n}$. This comes down to looking for rational functions f/g which are eigenvectors of ψ_q with eigenvalue 0 or 1. Can these be characterized?
3. There have been various improvements to the classical Chevalley-Warning theorem; see for example, [5] [20] [31] [23], and in particular, [2]. Are there analogous refinements for Corollary 4.2?

5 Remark on arithmetic mirror symmetry

There has been a lot of recent attention paid to studying similarities between zeta functions of mirror pairs of algebraic varieties. A selective list of references includes [4], [24], [19] [21] [8], [9], [12], [10], [14], [13], [35], [34], [11]. An alternate

viewpoint is to study the relation between the complements of these varieties within some fixed ambient spaces. The following is a brief remark in this direction.

Let X and Y be a strong mirror pair of Calabi-Yau varieties of dimension d defined over \mathbb{F}_q . We refer the reader to [35] for the definition of strong mirror pair. Fix projective embeddings $X \hookrightarrow \mathbb{P}^n$ and $Y \hookrightarrow \mathbb{P}^n$, and denote the complements of X and Y in their respective embeddings by V_X and V_Y . Let f and g be regular functions on V_X and V_Y , respectively. Define the L -function

$$L(V_X, f; T) := \exp \left(\sum_{m=1}^{\infty} S_m(V_X, f) \frac{T^m}{m} \right),$$

where

$$S_m(V_X, f) := \sum_{\bar{x} \in V_X(\mathbb{F}_{q^m})} \psi_m(f(\bar{x})).$$

Similarly, define $L(V_Y, g; T)$.

Question. Under suitable conditions on the characteristic p , is there a uniformizer π such that $L(V_X, f; T) \equiv L(V_Y, g; T) \pmod{\pi}$?

A heuristic proof is as follows. Theorems 3.1 and 4.1 suggest that there may exist a uniformizer π such that $L(V_X, f; T) \equiv Z(X, T)^{-1} \pmod{\pi}$ and $L(V_Y, g; T) \equiv Z(Y, T)^{-1} \pmod{\pi}$. The result would then follow from a conjecture of Wan [35], which implies that the zeta functions of strong mirror pairs satisfy $Z(X, T) \equiv Z(Y, T) \pmod{q}$.

As Wan's conjecture has been proven in the case of the Dwork family of hypersurfaces and its mirror (see [35]), this question can likely be proven in this case.

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